



# $B \rightarrow D^{**}$ semileptonic decay in covariant quark models à la Bakamjian Thomas

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February 1, 2008

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## Abstract

Once chosen the dynamics in one frame, for example the rest frame, the Bakamjian and Thomas method allows to define relativistic quark models in any frame. These models have been shown to provide, in the heavy quark limit, fully covariant current form factors as matrix elements of the quark current operator. They also verify the Isgur-Wise scaling and give a slope parameter  $\rho^2 > 3/4$  for all the possible choices of the dynamics. In this paper we study the  $L = 1$  excited states and derive the general formula, valid for any dynamics, for the scaling invariant form factors  $\tau_{1/2}^{(n)}(w)$  and  $\tau_{3/2}^{(n)}(w)$ . We also check the Bjorken-Isgur-Wise sum rule already demonstrated elsewhere in this class of models.

LPTHE Orsay-96/12  
PCCF RI 9601  
hep-ph/9605206

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# 1 Introduction

It was recently noticed [1] that it is possible, in the heavy mass limit, to formulate fully covariant quark models for form factors in which the current acts in the standard way on the heavy quark while the other quarks remain spectators. Following Bakamjian and Thomas (BT) [2], given the wave function in, say, the rest frame<sup>2</sup>, the hadron wave functions are defined in any frame through a unitary transformation, in such a way that Poincaré algebra is satisfied. It was shown in [1] that the  $\rho^2$  Isgur-Wise slope parameter was bounded in this class of model:  $\rho^2 > 0.75$ .

It is known that quark models are of special value when excited states are considered, since then no other hadronic method is available: lattice simulation as well as QCD sum rules are practically restricted to ground states because they use euclidean continuation. It is therefore tempting to apply this covariant approach to the  $B \rightarrow D^{**}$  decays. In [3] it has been shown that these covariant quark models satisfy the Bjorken sum rule [4], [5]. This is not a trivial achievement. It comes in this class of models because the boost of the wave functions is a unitary transformation that keeps the closure property of the Hilbert space in all frames.

In [3] the precise formulae for the  $\tau_{1/2}^{(n)}(w)$  and  $\tau_{3/2}^{(n)}(w)$  have not been computed except for  $w = 1$ . In view of practical phenomenology of the  $B \rightarrow D^{**}$  decays,  $\tau_{1/2}^{(n)}(w)$  and  $\tau_{3/2}^{(n)}(w)$  are obviously needed for any  $w$ . This is our main goal in the present paper. We do not want here to make a choice for our preferred internal wave function. These formulae may be used by anyone who wants to apply the BT method in the heavy mass limit to its preferred set of rest-frame, or infinite momentum frame or whichever frame wave functions. The Ansatz [6], although not explicitly of the BT type and computed in particular frames, would probably be obtained by the BT method with the harmonic oscillators. The same comment applies to the calculation of  $B \rightarrow D^{**}$  [7]. In deriving the formulae for  $\tau_{1/2}^{(n)}(w)$  and  $\tau_{3/2}^{(n)}(w)$ , we will also try to repeat, as much in a transparent way as possible, the content of the BT method applied to the heavy quark limit. This will be done in section 2. In section 3 we will compute the  $\tau_{1/2}^{(n)}(w)$  and  $\tau_{3/2}^{(n)}(w)$ . In section 4 we will derive directly from the latter formulae the Bjorken-Isgur-Wise sum rule.

## 2 Covariant quark models of form factors in the heavy mass limit

### 2.1 Framework

The main purpose of this model is to provide a way to implement covariance in the calculation of form factors. These form factors appear through matrix elements relating the initial and final states of the hadrons. We assume here a spectator quark model, that is, of all the quarks building the hadron, only one (labeled as 1) is the active particle. With this hypothesis, the amplitude of any process can be written as:

$$\langle \Psi' | O | \Psi \rangle = \int \frac{d\vec{p}'_1}{(2\pi)^3} \frac{d\vec{p}_1}{(2\pi)^3} \frac{d\vec{p}_2}{(2\pi)^3} \sum_{s'_1, s_1, s_2} \Psi_{s'_1, s_2}(\vec{p}'_1, \vec{p}_2)^* O(\vec{p}'_1, \vec{p}_1)_{s'_1, s_1} \Psi_{s_1, s_2}(\vec{p}_1, \vec{p}_2)$$

where we have used the so-called one particle variables, momentum  $\vec{p}_i$  and spin  $\vec{S}_i$ , and where  $O(\vec{p}'_1, \vec{p}_1)_{s'_1, s_1}$  is the matrix element of the *free* one-particle operator  $O$  between one particle states of the form  $|\vec{p}, s\rangle$  (we only take into account two quarks since we are dealing with mesons; the generalization to  $n$  particles is straightforward [1]).

Now, the covariance is introduced by expressing the relativistic 2-particle bound states  $\Psi_{s_1, s_2}(\vec{p}_1, \vec{p}_2)$  as a representation of the full Poincaré group, following the Bakamjian-Thomas formalism. It turns out that this process is made easier if we change the variables that characterize the state  $\Psi$ , that is, if we introduce the total momentum  $\vec{P}$ , ( $\vec{P} = \vec{p}_1 + \vec{p}_2$ ), the internal momenta  $\vec{k}_1, \vec{k}_2$ , ( $\vec{k}_1 + \vec{k}_2 = \vec{0}$ ), and the internal spins  $\vec{S}'_1, \vec{S}'_2$ , leading to another wave function  $\Psi_{s_1, s_2}^{int}(\vec{P}, \vec{k}_2)$  which is related to the previous one by the *unitary* transformation:

$$\Psi_{s_1, s_2}(\vec{p}_1, \vec{p}_2) = \sqrt{\frac{p_1^o + p_2^o}{M_o}} \frac{\sqrt{k_1^o k_2^o}}{\sqrt{p_1^o p_2^o}} \sum_{s'_1, s'_2} (D(\mathbf{R}_1)_{s_1, s'_1} D(\mathbf{R}_2)_{s_2, s'_2}) \Psi_{s'_1, s'_2}^{int}(\vec{P}, \vec{k}_2) \quad (2.1)$$

Let us explain the notation used in this last equation:

- $p_i^o = \sqrt{\vec{p}_i^2 + m_i^2}$

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<sup>2</sup>Any starting frame can be chosen, for example the infinite momentum frame although some care is then needed.

- $M_o = \sqrt{(\Sigma p_j)^2}$
- $k_2 = B_{\Sigma p_j}^{-1} p_2$
- $D(\mathbf{R}_i)_{s_i, s'_i}$  is the  $(s_i, s'_i)$  matrix element for a spin 1/2 particle of the Wigner rotation  $\mathbf{R}_i$ , which describes the connection spin $\leftrightarrow$ internal spin and which is given by:

$$\mathbf{R}_i = B_{p_i}^{-1} B_{\Sigma p_j} B_{k_i} \quad (2.2)$$

where  $B_p$  is the Lorentz boost  $(\sqrt{p^2}, \vec{0}) \rightarrow p$ .

Note that there is no apparent dependence upon  $\vec{k}_1$  owing to the relation which defines the internal momenta.

Returning now to the Poincaré group transformation laws, their generators are defined according to:

$$\left\{ \begin{array}{l} \text{space translations : } \vec{P} \\ \text{time translations : } H = P^o = \sqrt{\vec{P}^2 + M^2} \\ \text{rotations : } \vec{J} = -i\vec{P} \times \frac{\partial}{\partial \vec{P}} + \vec{S} \\ \text{boosts : } \vec{K} = -\frac{i}{2} \left\{ P^o, \frac{\partial}{\partial \vec{P}} \right\} - \frac{\vec{P} \times \vec{S}}{P^o + M} \end{array} \right. \quad (2.3)$$

where  $\vec{S} = (\vec{S}'_1 + \vec{S}'_2) - i\vec{k}_2 \times \frac{\partial}{\partial \vec{k}_2}$  is the spin of the meson

These generators depend on the interaction through the mass operator  $M$ . Any given quark model corresponds to one choice of the mass operator  $M$ , in order, for example, to describe the mass spectrum at best. In this paper we do not want to enter into the discussion of the best choice for the latter operator. We simply need  $M$  to depend only on the internal variables and to be invariant by rotation in order to obey Poincaré algebra. The mass  $M_0 = \sqrt{(\Sigma p_j)^2}$  introduced above is not the real mass. It is just a tool to define properly the unitary change of variables (2.1). Only in the limit when the quark interaction would vanish, would  $M_0$  become equal to  $M$ . In such a limit, the generators in (2.3) would be the standard Poincaré generators for the considered set of free particles. As such it would naturally obey Poincaré algebra. In some sense, the trick used above in order to have the correct Poincaré algebra even when interaction is present is to mimic, so to say, the behavior of free particles by the definition of  $p_i^0$ ,  $M_0$ . This does not mean that we make a weak interaction approximation. One may say that the change of variables (2.1) is inspired from the non-interacting case, but it may be applied whatever the strength of the interaction is.

It is then possible to construct [1] the wave function of the bound state of quarks moving with total momentum  $\vec{P}$  and we get:

$$\Psi_{s_1, s_2}^{int}(\vec{P}, \vec{k}_2) = (2\pi)^3 \delta(\vec{p}_1 + \vec{p}_2 - \vec{P}) \varphi_{s_1, s_2}(\vec{k}_2) \quad (2.4)$$

where  $\varphi_{s_1, s_2}(\vec{k}_2)$  is an eigenstate of  $M$ ,  $\vec{S}$  and  $S_z$ , so that the relativistic wave function expressed with the momenta  $\vec{p}_i$  and spins  $\vec{S}_i$  reads:

$$\Psi_{s_1, s_2}(\vec{p}_1, \vec{p}_2) = \sqrt{\frac{p_1^o + p_2^o}{M_o}} \frac{\sqrt{k_1^o k_2^o}}{\sqrt{p_1^o p_2^o}} \sum_{s'_1, s'_2} D(\mathbf{R}_1)_{s_1, s'_1} D(\mathbf{R}_2)_{s_2, s'_2} (2\pi)^3 \delta(\vec{p}_1 + \vec{p}_2 - \vec{P}) \varphi_{s'_1, s'_2}(\vec{k}_2)$$

Notice that for  $\vec{P} = \vec{0}$ , this last formula gives

$$\Psi_{s_1, s_2}^{(\vec{P}=\vec{0})}(\vec{p}_1, \vec{p}_2) = (2\pi)^3 \delta(\vec{p}_1 + \vec{p}_2) \varphi_{s_1, s_2}(\vec{p}_2)$$

which just expresses the fact that  $\varphi$  is the rest-frame internal wave function.

Putting all these things together, we get the new following workable relation:

$$\begin{aligned} \langle \vec{P}' | O | \vec{P} \rangle = & \int \frac{d\vec{p}_2}{(2\pi)^3} \sqrt{\frac{(p_1'^o + p_2^o)(p_1^o + p_2^o)}{M_o' M_o}} \frac{\sqrt{k_1'^o k_1^o}}{\sqrt{p_1'^o p_1^o}} \frac{\sqrt{k_2'^o k_2^o}}{\sqrt{p_2'^o p_2^o}} \\ & \sum_{s_1', s_2'} \sum_{s_1, s_2} \varphi'_{s_1', s_2'}(\vec{k}_2)^* [D(\mathbf{R}_1'^{-1}) O(\vec{p}_1', \vec{p}_1) D(\mathbf{R}_1)]_{s_1', s_1} D(\mathbf{R}_2'^{-1} \mathbf{R}_2)_{s_2', s_2} \varphi_{s_1, s_2}(\vec{k}_2) \quad (2.5) \end{aligned}$$

since  $\vec{p}_2' = \vec{p}_2$  for the spectator quark. At this stage, this matrix element is not covariant in general, because the current operator  $O$  is not covariant with respect to the transformations (2.3) which contain the interaction. However, in the limit where the masses  $m_1$  and  $m_1'$  of the quark 1 tend to infinity, this last equation becomes fully covariant as shown in [1].

So, up to now, we have replaced the knowing of a relativistic wave function of a moving bound state by the knowing of a rest-frame wave function, which is an eigenstate of a mass operator (in this case the hamiltonian) which does not have to be relativistic. Of course, the physics lies in the complete determination of those rest-functions  $\varphi_{s_1, s_2}(\vec{k}_2)$ , and in the expression of the current operator. Another point which must be emphasized is the unitary nature of the transformations which have been used. Unitarity ensures that, in any frame, all sets of wave functions are orthonormal and satisfy the closure relation. This will prove to be essential for the derivation of the Bjorken sum rule (see section 4).

From now on, we will assume that the quark 1 has an infinite mass to enable the Heavy Quark Symmetries (flavor and spin symmetries) and the covariance of the matrix elements. Besides, we will denote the internal spins  $\vec{s}_i$  and no longer  $\vec{S}_i'$ .

## 2.2 The wave functions

In the rest frame ( $\vec{P} = \vec{0}$ ), the generator of rotations writes:

$$\vec{J} = \vec{s}_1 + \vec{j}. \quad (2.6)$$

In quark models,  $\vec{j}$  equals  $\vec{s}_2 + \vec{l}$  and  $\vec{l} = -i\vec{k}_2 \times \partial/\partial\vec{k}_2$  is the relative orbital angular momentum of the second quark, which is equal to 1 in the case of  $D^{**}$  mesons. However, beyond quark models, the decomposition (2.6) is very useful in general in the heavy quark limit. Indeed, according to Heavy Quark Symmetry, the spin  $\vec{s}_1$  of the heavy quark is conserved, i.e. it commutes with the Hamiltonian. It then results from conservation of the total angular momentum  $\vec{J}$  that the angular momentum  $\vec{j}$  is also conserved. It is then natural to label the lightest parity-even states ( $l = 1$ ,  $P$ -wave in the quark models), according to the values of  $j$ , namely  $j = 1/2$  and  $j = 3/2$ . It is known [5] that all hadrons within  $j = 1/2$ , respectively  $j = 3/2$ , multiplet, are related by the heavy quark symmetry<sup>3</sup>. Finally, when combined with the heavy quark spin  $s_1$ , we get two distinct multiplets: one with  $j = 1/2$  and  $J^P = 0^+$  or  $1^+$ , and the other with  $j = 3/2$  and  $J^P = 1^+$  or  $2^+$ . The corresponding states will be denoted respectively, using the notation  $|jJ^P\rangle$ :

$$|\tfrac{1}{2} 0^+\rangle \quad |\tfrac{1}{2} 1^+\rangle \quad |\tfrac{3}{2} 1^+\rangle \quad |\tfrac{3}{2} 2^+\rangle$$

### 2.2.1 Generic form of the $\varphi$ 's

How are we going to write the rest-frame internal wave functions? Recall that they are eigenstates of the mass operator  $M$ . In the model, this operator is assumed to be rotationally invariant, to depend only upon the internal variables and, of course, to conserve parity. So it commutes with  $\vec{J}$ , which in turn commutes with the hamiltonian  $H$ . Now, the heavy quark spin symmetry implies the invariance with respect to  $\vec{s}_1$  of the system, that is  $[H, \vec{s}_1] \equiv [M, \vec{s}_1] = 0$ . As a consequence,  $[H, \vec{j}] \equiv [M, \vec{j}] = 0$  and the total Hilbert space  $\mathcal{H}$  of the  $\varphi$ 's can be factorized as a tensorial product of a spin space  $\mathcal{H}_{\vec{s}_1}$ , related to the spin  $\vec{s}_1$ , and of a "spin-orbit" space  $\mathcal{H}_{\vec{j}}$  which describes the internal spin  $\vec{s}_2$  of the light quark and the relative orbital angular momentum  $\vec{l}$ . We must therefore build a representation of  $\vec{j} = \vec{s}_2 + \vec{l}$ . But we know a priori nothing about  $[H, \vec{l}]$  or  $[H, \vec{s}_2]$  so we cannot reproduce the same kind of argument used for the

<sup>3</sup>All these properties are valid in the model, and we shall use them in the next subsection.

decomposition of  $\mathcal{H}$  into  $\mathcal{H}_{\vec{s}_1} \otimes \mathcal{H}_{\vec{j}}$ . Yet, we are concerned by the two lightest parity even multiplets ( $j = 1/2$  and  $j = 3/2$ ), with parity  $+1$ . They all correspond to the same value of the orbital momentum,  $l = 1$ . Indeed, due to parity conservation,  $l = 1$  cannot be mixed with  $l = 0, 2$ , and due to the conservation of  $\vec{j}$ , it cannot be mixed with  $l = 3, 5$  which cannot produce  $j = 1/2, 3/2$  when combined with  $s_2 = 1/2$ . Whence, the only way of constructing this representation of  $\vec{j}$  is  $\vec{l} \otimes \vec{s}_2$  with  $l = 1$ .

However, we must stress again that the radial part of the wave functions and the energies depend on  $j$ . The four  $l = 1, s_2 = 1/2$  states are not degenerate in energy due to an  $\vec{l} \cdot \vec{s}_2$  force which does not vanish in the heavy quark limit. Since the eigenvalue of  $\vec{l} \cdot \vec{s}_2$  is equal to  $(j(j+1) - l(l+1) - s_2(s_2+1))/2 = j(j+1)/2 - 1 - 3/8$ , we see that the four states will split as expected into two multiplets corresponding to  $j = 1/2$  and  $j = 3/2$  respectively. The spin and orbital parts of the wave functions combine through Clebsch-Gordan coefficients to build up the  $j = 1/2, 3/2$  eigenstates. As a consequence the  $^3P_1$  and the  $^1P_1$  states will mix to produce the  $j$  eigenstates. Finally, we can factorize the wave function:

$$\Psi_{M, s_1, s_2}^{j, J^P}(\vec{p}) = (j J^P_{M, s_1, s_2})(\hat{p}) \sqrt{\frac{4\pi}{3}} \|\vec{p}\| \phi_j(\|\vec{p}\|^2) \quad (2.7)$$

where

$$(j J^P_{M, s_1, s_2})(\hat{p}) = \sum_m \langle j \ m \ 1/2 \ M - m \mid J \ M \rangle \chi_{s_1}^{M-m} \sum_{m'} \langle 1 \ m' \ 1/2 \ m - m' \mid j \ m \rangle \chi_{s_2}^{m-m'} Y_1^{m'}(\hat{p}) \quad (2.8)$$

$Y_l^m(\hat{p})$  being the spherical harmonics  $\langle \theta, \phi \mid lm \rangle$  and  $\phi_j(\|\vec{p}\|^2)$  a radial function which remains to be determined, and is normalized according to

$$\int \frac{d\vec{p}}{(2\pi)^3} \frac{p^2}{3} |\phi_j(\|\vec{p}\|^2)|^2 = 1.$$

The unusual writing of the radial part in (2.7) is a matter of convenience, meant to simplify further calculation. Moreover,  $\chi_s^m$  is the column matrix defined by:

$$\chi^{+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi^{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

From now on, we will often identify the indices  $1/2$  with 1 and  $-1/2$  with 2 as a way of spotting the matrix elements which are related to spin. Consequently, the  $\chi_s^m$  equal  $\delta_{ms}$ .

Let us concentrate for awhile on the spin-orbital part of the wave function. The calculations are made easier if we deal with another composition of  $\vec{l}$ ,  $\vec{s}_1$  and  $\vec{s}_2$  in order to get  $\vec{J}$ , that is if, instead of writing  $\vec{s}_1 \otimes (\vec{s}_2 \otimes \vec{l})$ , we use  $(\vec{s}_1 \otimes \vec{s}_2) \otimes \vec{l}$ . With that decomposition, we obtain the so-called  $|^{2S+1}P_J\rangle$  states, which can be related to the  $|j J^P\rangle$  through the “6j” coefficients according to:

$$|j J^P\rangle = \sum_S (-)^{s_1+s_2+l+J} \sqrt{(2S+1)(2j+1)} \left\{ \begin{matrix} s_1 & s_2 & S \\ l & J & j \end{matrix} \right\} |^{2S+1}P_J\rangle$$

which reads in the present case

$$\left\{ \begin{array}{l} |\frac{1}{2} 0^+\rangle = |^3P_0\rangle \\ |\frac{1}{2} 1^+\rangle = -\frac{1}{\sqrt{3}} |^1P_1\rangle + \sqrt{\frac{2}{3}} |^3P_1\rangle \\ |\frac{3}{2} 1^+\rangle = \sqrt{\frac{2}{3}} |^1P_1\rangle + \frac{1}{\sqrt{3}} |^3P_1\rangle \\ |\frac{3}{2} 2^+\rangle = |^3P_2\rangle \end{array} \right.$$

with

$$\begin{aligned}
({}^{2S+1}P_{J_{s_1, s_2, M}})(\hat{p}) &= \sum_m \langle 1 \ m \ S \ M-m \mid J \ M \rangle Y_1^m(\hat{p}) \sum_{a, b} \langle 1/2 \ a \ 1/2 \ b \mid S \ M-m \rangle (\chi^a)_{s_1} (\chi^b)_{s_2} \\
&= \sum_m \langle 1 \ m \ S \ M-m \mid J \ M \rangle \langle 1/2 \ s_1 \ 1/2 \ s_2 \mid S \ M-m \rangle Y_1^m(\hat{p})
\end{aligned}$$

$$\text{since } (\chi^m)_s = \delta_{m s}$$

It may also be useful to remark that the Clebsch-Gordan coefficient  $\langle 1/2 \ s_1 \ 1/2 \ s_2 \mid S \ m \rangle$  can be written as the matrix element of a combination of Pauli matrices according to:

$$\begin{aligned}
\langle 1/2 \ s_1 \ 1/2 \ s_2 \mid 0 \ m \rangle &= \frac{i}{\sqrt{2}} (\sigma_2)_{s_1 s_2} \\
\langle 1/2 \ s_1 \ 1/2 \ s_2 \mid 1 \ m \rangle &= \frac{i}{\sqrt{2}} \left[ \left( \vec{e}^{(m)} \cdot \vec{\sigma} \right) \sigma_2 \right]_{s_1 s_2}
\end{aligned} \tag{2.9}$$

where we have used the standard basis:

$$\begin{cases} \vec{e}^{(+1)} = -\frac{1}{\sqrt{2}} (\vec{e}_x + i \vec{e}_y) \\ \vec{e}^{(0)} = \vec{e}_z \\ \vec{e}^{(-1)} = \frac{1}{\sqrt{2}} (\vec{e}_x - i \vec{e}_y) \end{cases}$$

(The demonstration of these relations can be carried out by evaluating the right hand sides of (2.9) and then by comparing the results with the values of the Clebsch-Gordan coefficients).

Finally, by substituting the expressions of the  $Y_l^m(\hat{p})$  in spherical coordinates, the  $|{}^{2S+1}P_J\rangle$  spin-orbital wave functions write:

$$\begin{cases} ({}^3P_{0, s_1, s_2})(\hat{p}) = -\frac{i}{\sqrt{8\pi}} [(\vec{\sigma} \cdot \vec{p}) \sigma_2]_{s_1 s_2} \cdot \frac{1}{\|\vec{p}\|} \\ ({}^1P_{1, s_1, s_2, m})(\hat{p}) = i\sqrt{\frac{3}{8\pi}} \left[ \left( \vec{e}^{(m)} \cdot \vec{p} \right) \sigma_2 \right]_{s_1 s_2} \cdot \frac{1}{\|\vec{p}\|} \\ ({}^3P_{1, s_1, s_2, m})(\hat{p}) = \sqrt{\frac{3}{16\pi}} \left[ \left( \vec{e}^{(m)} \cdot (\vec{p} \wedge \vec{\sigma}) \right) \sigma_2 \right]_{s_1 s_2} \cdot \frac{1}{\|\vec{p}\|} \\ ({}^3P_{2, s_1, s_2, m})(\hat{p}) = -i\sqrt{\frac{3}{8\pi}} \left[ \left( \sigma^i p^j e_{ij}^{(m)} \right) \sigma_2 \right]_{s_1 s_2} \cdot \frac{1}{\|\vec{p}\|} \end{cases}$$

where  $e_{ij}^m$  is the rest-frame polarization tensor, that is a symmetrical tensor with vanishing spur.

We are now able to go one step further into the determination of the transition amplitude  $\langle \vec{P}' | O | \vec{P} \rangle$ . In the present case, the state  $|\vec{P}\rangle$  represents a pseudoscalar state  ${}^1S_0$  and the state  $|\vec{P}'\rangle$  one of the  $({}^{2S+1}P_J)_j$  states described above. In other words, the  $\varphi_{s_1, s_2}(\vec{k})$ 's take the form:

$$\begin{aligned}
\varphi_{s_1, s_2}(\vec{k}_2) &= \frac{i}{\sqrt{2}} (\sigma_2)_{s_1 s_2} \varphi(\|\vec{k}_2\|^2) \\
\varphi'_{s'_1, s'_2}(\vec{k}'_2) &= \frac{i}{\sqrt{2}} (\chi \sigma_2)_{s'_1 s'_2} \phi_j(\|\vec{k}'_2\|^2)
\end{aligned}$$

where

$$\chi_{(2S+1)P_J} = \begin{cases} -\frac{1}{\sqrt{3}} (\vec{\sigma} \cdot \vec{k}'_2) & \text{for the } {}^3P_0 \text{ state} \\ (\vec{e}^{(m)} \cdot \vec{k}'_2) & \text{for the } {}^1P_1 \text{ state} \\ -\frac{i}{\sqrt{2}} \left[ \vec{e}^{(m)} \cdot (\vec{k}'_2 \wedge \vec{\sigma}) \right] & \text{for the } {}^3P_1 \text{ state} \\ -(\sigma^i k_2'^j e_{ij}^{(m)}) & \text{for the } {}^3P_2 \text{ state} \end{cases}$$

Then, by substituting these expressions into (2.5) and using the relation  $\sigma_2 D(\mathbf{R}) \sigma_2 = D(\mathbf{R}^{-1})^t$ , we get:

$${}_j \langle {}^{2S+1}P_J | O | {}^1S_0 \rangle = \int \frac{d\vec{p}_2}{(2\pi)^3} \sqrt{\frac{(p_1'^o + p_2^o)(p_1^o + p_2^o)}{M_o' M_o}} \frac{\sqrt{k_1'^o k_1^o}}{\sqrt{p_1'^o p_1^o}} \frac{\sqrt{k_2'^o k_2^o}}{\sqrt{p_2^o p_2^o}} \\ \times \frac{1}{2} \text{Tr} \left[ \chi^\dagger D(\mathbf{R}_1'^{-1}) O(\vec{p}_1', \vec{p}_1) D(\mathbf{R}_1) D(\mathbf{R}_2^{-1} \mathbf{R}_2') \right] \phi_j(\|\vec{k}_2'\|^2)^* \varphi(\|\vec{k}_2\|^2) \quad (2.10)$$

### 2.3 Switching to Dirac notation

As already stated, it can be shown directly on (2.5) that the current matrix element is covariant in the heavy mass limit  $m_1 \rightarrow \infty$ . We shall demonstrate covariance by changing to Dirac notation, in which the current matrix elements will eventually appear as *manifestly covariant*. This notation will also be very useful to handle later calculations. The idea is to insert the  $2 \times 2$  matrices, that appeared in the precedent sections, into the  $2 \times 2$  upper left block of a  $4 \times 4$  matrix which is then completed with zeros:  $\begin{pmatrix} \chi & 0 \\ 0 & 0 \end{pmatrix}$ . This leads to the following formula:

$${}_j \langle {}^{2S+1}P_J | O | {}^1S_0 \rangle = \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^o} \frac{\sqrt{u_o' u_o}}{p_1^o p_1'^o} \frac{k_1^o}{\sqrt{k_1^o + m_1}} \frac{k_2^o}{\sqrt{k_2^o + m_2}} \frac{k_1'^o}{\sqrt{k_1'^o + m_1'}} \frac{k_2'^o}{\sqrt{k_2'^o + m_2}} \\ \times \frac{1}{16} \text{Tr} \left\{ O(m_1 + \not{p}_1) (1 + \not{u}) (m_2 + \not{p}_2) (1 + \not{u}') \left( \mathbf{B}_u \chi'^\dagger \mathbf{B}_u^{-1} \right) (m_1' + \not{p}_1') \right\} \phi_j(\|\vec{k}_2'\|^2)^* \varphi(\|\vec{k}_2\|^2) \quad (2.11)$$

where  $u$  and  $u'$  are defined by

$$M_o u = p_1 + p_2 \quad (u^2 = 1) \qquad M_o' u' = p_1' + p_2 \quad (u'^2 = 1)$$

The derivation of (2.11) is quite straightforward: it stems from

- the expressions of the Wigner rotation matrices as a product of three boosts, see equation (2.2)
- the expression of the matrix  $O(\vec{p}', \vec{p})$  as [1]

$$O(\vec{p}', \vec{p}) = \frac{\sqrt{m_1' m_1}}{\sqrt{p_1'^0 p_1^0}} \frac{1 + \gamma^0}{2} \mathbf{B}_{p_1'}^{-1} O \mathbf{B}_{p_1} \frac{1 + \gamma^0}{2}$$

In this last relation, the boosts  $\mathbf{B}_p$  are written in the Dirac representation according to

$$\mathbf{B}_p = \frac{m + \not{p} \gamma^0}{\sqrt{2m(p^0 + m)}}$$

- the properties of  $\mathbf{B}_u$ , that is, for instance:

$$\mathbf{B}_u(1, \vec{0}) = u \qquad (\text{idem for } \mathbf{B}_{u'}) \\ p_2 = \mathbf{B}_u k_2 = \mathbf{B}_{u'} k_2'$$

- the following formula, which will also be used throughout the rest of this paper in order to evaluate the terms  $\mathbf{B}_{u'} \chi^\dagger \mathbf{B}_{u'}^{-1}$ :

$$\mathbf{B}_u(\gamma_\mu x^\mu) \mathbf{B}_u^{-1} = \gamma_\mu (\mathbf{B}_u x)^\mu \quad (2.12)$$

(the  $\gamma^\mu$  are forming a 4-vector), and which gives for example:

$$\mathbf{B}_u \gamma_o \mathbf{B}_u^{-1} = \mathbf{B}_u(1, \vec{0}) \gamma^\mu \mathbf{B}_u^{-1} = \gamma^\mu \cdot \mathbf{B}_u(1, \vec{0}) = \not{u}$$

- the fact that the Wigner rotation matrices, the  $\chi$ 's and  $\gamma_o$  are all block-diagonal matrices, implying that they commute with the matrix  $\gamma_o$  which has scalar blocks.



## 2.4 The heavy-mass limit

Let us consider now the heavy-mass limit of (2.11). By doing this, we mean that we take  $m_1, m'_1 \longrightarrow +\infty$  and, at the same time, we keep fixed the following ratios:

$$v' = \frac{P'}{M'} \qquad v = \frac{P}{M}$$

Therefore, we have:

$$\begin{array}{ll} \frac{p_1}{m_1} \longrightarrow v & \frac{p'_1}{m'_1} \longrightarrow v' \\ u \longrightarrow v & u' \longrightarrow v' \\ \frac{k_1^o}{m_1} \longrightarrow 1 & \frac{k_1'^o}{m'_1} \longrightarrow 1 \\ k_2 \longrightarrow \mathbf{B}_v^{-1} p_2 & k'_2 \longrightarrow \mathbf{B}_{v'}^{-1} p_2 \end{array}$$

As a consequence, since the scalar product of 4-vectors is invariant, we get:

$$(\mathbf{B}_v^{-1} p_2)^0 = p_2 \cdot v \qquad (\mathbf{B}_{v'}^{-1} p_2)^0 = p_2 \cdot v'$$

We deduce from these formulae the invariance of  $\|\overrightarrow{\mathbf{B}_v^{-1} p_2}\|^2$  and  $\|\overrightarrow{\mathbf{B}_{v'}^{-1} p_2}\|^2$ :

$$\|\overrightarrow{\mathbf{B}_v^{-1} p_2}\|^2 = (p_2 \cdot v)^2 - m_2^2, \qquad \|\overrightarrow{\mathbf{B}_{v'}^{-1} p_2}\|^2 = (p_2 \cdot v')^2 - m_2^2 \quad (2.13)$$

Finally, the heavy mass limit of the relation (2.11) reads:

$$\begin{aligned} {}_j \langle {}^{2S+1}P_J | O | {}^1S_0 \rangle &= \frac{1}{8} \frac{1}{\sqrt{v_o v'_o}} \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^o} \frac{\sqrt{(p_2 \cdot v')(p_2 \cdot v)}}{\sqrt{(p_2 \cdot v' + m_2)(p_2 \cdot v + m_2)}} \\ &\quad \times \text{Tr} \left\{ O (1 + \not{p}) (m_2 + \not{p}_2) (\mathbf{B}_{v'} \chi^\dagger \mathbf{B}_{v'}^{-1}) (1 + \not{p}') \right\} \phi_j(\|\overrightarrow{\mathbf{B}_{v'}^{-1} p_2}\|^2)^* \varphi(\|\overrightarrow{\mathbf{B}_v^{-1} p_2}\|^2) \end{aligned} \quad (2.14)$$

(2.14) is the starting point for the calculation of the transition amplitudes from a pseudoscalar state towards the  $({}^{2S+1}P_J)_j$  states ( $D^{**}$  states): that is the purpose of the next section.

It is important here to stress that the expression (2.14) is covariant. The integration measure  $d\vec{p}_2/p_2^o$  is invariant; so is the trace as obvious from expressions (3.1) for  $\mathbf{B}_{v'} \chi^\dagger \mathbf{B}_{v'}^{-1}$  and finally the arguments of the wave functions are invariant from (2.13).

## 3 Transition amplitudes and Isgur-Wise scaling

### 3.1 Preliminary calculation

Before calculating the spur which appears in (2.14), we must evaluate the  $\mathbf{B}_{v'} \chi^\dagger \mathbf{B}_{v'}^{-1}$  terms, for each  $\chi_{(2S+1)P_J}$ . The procedure is quite automatic:

1. We have to write each  $\chi$ , in the rest frame, in a covariant way, that is using 4-vectors instead of 3-vectors. That is realized by introducing the 4-vector  $n^\mu = (1, \vec{0})$  and the expression of the Pauli matrices in term of the Dirac matrices:

- ( ${}^3P_0$ ): Starting from  $\chi_{({}^3P_0)}$ , we get the  $4 \times 4$  matrix in the following way:

$$\chi_{({}^3P_0)}^\dagger = -\frac{1}{\sqrt{3}} (\vec{\sigma} \cdot \vec{k}'_2)^\dagger = -\frac{1}{\sqrt{3}} (\vec{\sigma} \cdot \vec{k}'_2) \rightsquigarrow -\frac{1}{\sqrt{3}} \underbrace{\gamma_5 \gamma_0 (\vec{\gamma} \cdot \vec{k}'_2)}_{\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}} \frac{1 + \gamma_0}{2} = \frac{1}{\sqrt{3}} \gamma_5 (\vec{\gamma} \cdot \vec{k}'_2) \frac{1 + \gamma_0}{2}$$

where the  $\frac{1+\gamma_0}{2}$  factor in the right hand side gives the form  $\begin{pmatrix} \chi & 0 \\ 0 & 0 \end{pmatrix}$  to the  $\chi$ . By introducing the  $n^\mu$  mentioned above, this can also be written as:

$$\chi_{(3P_0)}^\dagger = \frac{1}{\sqrt{3}} \gamma_5 \left[ -\not{k}'_2 + k'^0_2 \gamma_0 \right] \frac{1+\gamma_0}{2} = \frac{1}{\sqrt{3}} \gamma_5 \left[ -\not{k}'_2 + k'^0_2 \gamma_\mu n^\mu \right] \frac{1+\gamma_0}{2}$$

that is, finally,

$$\chi_{(3P_0)}^\dagger = -\frac{1}{\sqrt{3}} \gamma_5 \left[ \not{k}'_2 - k'^0_2 \not{n} \right] \frac{1+\gamma_0}{2}$$

- ( $^1P_1$ ): Introducing the 4-vector  $e = (0, \vec{e})$  (we drop the polarization index  $m$  for the time being) such that  $e \cdot n = 0$ , it is not difficult to obtain:

$$\chi_{(^1P_1)}^\dagger = -(k'_2 \cdot e^*) \frac{1+\gamma_0}{2}$$

- ( $^3P_1$ ): Another manipulation is performed here in order to deal with the cross product, but the idea is still the same: to add a fourth component.

$$\left[ \vec{e} \cdot (\vec{k}'_2 \wedge \vec{\sigma}) \right]^\dagger = \epsilon^{ijk} e_i^* k'_{2j} \sigma_k \rightsquigarrow -\epsilon^{ijk} e_i^* k'_{2j} \gamma_5 \gamma_k \gamma_0 \rightsquigarrow -\epsilon^{\mu\nu\rho\sigma} n_\mu e_\nu^* k'_{2\rho} \gamma_5 \gamma_\sigma \gamma_0$$

Then, we get

$$\chi_{(^3P_1)}^\dagger = -\frac{i}{\sqrt{2}} \epsilon^{\mu\nu\rho\sigma} n_\mu e_\nu^* k'_{2\rho} \gamma_\sigma \gamma_5 \frac{1+\gamma_0}{2}$$

- ( $^3P_2$ ): In this last case, we generalize the polarization tensor  $e_{ij}$  into  $e_{\mu\nu}$  which is symmetrical and has a vanishing spur and obeys  $n^\mu e_{\mu\nu} = 0$ . Thanks to this last property, the  $\chi$  writes:

$$\chi_{(^3P_2)}^\dagger = \gamma_5 \gamma^\mu k'^\nu_2 e_{\mu\nu}^* \frac{1+\gamma_0}{2}$$

2. We now must evaluate the expressions  $\mathbf{B}_{v'} \chi^\dagger \mathbf{B}_{v'}^{-1}$ . That is achieved by inserting the factor  $\mathbf{B}_{v'} \mathbf{B}_{v'}^{-1}$  at the right places in the previous formulae and extensively using (2.12). Moreover, we introduce the following notations:

$$\epsilon_{\mu\nu} = \mathbf{B}_{v'} e_{\mu\nu} \quad \epsilon_\mu = \mathbf{B}_{v'} e_\mu$$

As a result, the  $\mathbf{B}_{v'} \chi^\dagger \mathbf{B}_{v'}^{-1}$  read:

$$\begin{cases} \mathbf{B}_{v'} \chi_{(3P_0)}^\dagger \mathbf{B}_{v'}^{-1} = \frac{1}{\sqrt{3}} [\not{p}_2 - (p_2 \cdot v') \not{v}'] \gamma_5 \frac{1+\not{v}'}{2} \\ \mathbf{B}_{v'} \chi_{(^1P_1)}^\dagger \mathbf{B}_{v'}^{-1} = -(p_2 \cdot e^*) \frac{1+\not{v}'}{2} \\ \mathbf{B}_{v'} \chi_{(^3P_1)}^\dagger \mathbf{B}_{v'}^{-1} = -\frac{i}{\sqrt{2}} \epsilon^{\mu\nu\rho\sigma} v'_\mu \epsilon_\nu^* p_{2\rho} \gamma_\sigma \gamma_5 \frac{1+\not{v}'}{2} \\ \mathbf{B}_{v'} \chi_{(^3P_2)}^\dagger \mathbf{B}_{v'}^{-1} = -\gamma^\mu p_2^\nu \epsilon_{\mu\nu}^* \gamma_5 \frac{1+\not{v}'}{2} \end{cases} \quad (3.1)$$

Notice the  $\frac{1+\not{v}'}{2}$  term in the previous relations; when contracted with the factor  $(1+\not{v}')$  of the spur in (2.14), the result will be  $(1+\not{v}')$  since  $\not{v}'^2 = 1$ . So, from now on, we will drop it from the expressions  $\mathbf{B}_{v'} \chi^\dagger \mathbf{B}_{v'}^{-1}$ .

## 3.2 The transition amplitudes

### 3.2.1 Definitions

On inserting the expressions (3.1) from the previous section into (2.14), three different kinds of integrals appear using the covariance of the matrix elements, (2.14). Therefore, we may introduce the following definitions regarding the result of these integrals:

$$A^{(j)} = \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{(p_2 \cdot v')(p_2 \cdot v)}}{\sqrt{(p_2 \cdot v' + m_2)(p_2 \cdot v + m_2)}} \phi_j(\|\overrightarrow{\mathbf{B}_{v'}^{-1} p_2}\|^2)^* \varphi(\|\overrightarrow{\mathbf{B}_v^{-1} p_2}\|^2)$$

$$\begin{aligned}
B^{(j)} v^\mu + B'^{(j)} v'^\mu &= \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{(p_2 \cdot v')(p_2 \cdot v)}}{\sqrt{(p_2 \cdot v' + m_2)(p_2 \cdot v + m_2)}} \phi_j(\|\vec{B}_{v'}^{-1} p_2\|^2)^* \varphi(\|\vec{B}_v^{-1} p_2\|^2) p_2^\mu \\
D_1^{(j)}(v^\mu v'^\nu + v^\nu v'^\mu) + D_2^{(j)} v^\mu v^\nu + D_3^{(j)} v'^\mu v'^\nu + D_4^{(j)} g^{\mu\nu} \\
&= \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{(p_2 \cdot v')(p_2 \cdot v)}}{\sqrt{(p_2 \cdot v' + m_2)(p_2 \cdot v + m_2)}} \phi_j(\|\vec{B}_{v'}^{-1} p_2\|^2)^* \varphi(\|\vec{B}_v^{-1} p_2\|^2) p_2^\mu p_2^\nu
\end{aligned}$$

where the  $A$ ,  $B$ 's and the  $D$ 's are function of  $w = v \cdot v'$ .

### 3.2.2 Results

We now have all the pieces to reduce (2.14) further for all the states  $(jJ^P)$ . Straightforward calculations and a little diracology lead to:

$$\begin{aligned}
\langle \tfrac{1}{2} 0^+ | O | 1 S_0 \rangle &= \frac{1}{8\sqrt{3}} \frac{1}{\sqrt{v_o v'_o}} \text{Tr} \{ O (1 + \not{v}) (1 - \not{v}') \gamma_5 \} \\
&\quad \times \left[ (1 + v \cdot v') \left( m_2 B^{(1/2)} + D_1^{(1/2)} + D_2^{(1/2)} \right) + 3 D_4^{(1/2)} \right] \\
\langle \tfrac{1}{2} 1^+ | O | 1 S_0 \rangle &= \frac{1}{8\sqrt{3}} \frac{1}{\sqrt{v_o v'_o}} \left( m_2 B^{(1/2)} + D_1^{(1/2)} + D_2^{(1/2)} \right) \\
&\quad \times \left[ \text{Tr} \{ O (1 + \not{v}) (1 + \not{v}') \} (v \cdot \epsilon^*) + i \epsilon^{\mu\nu\rho\sigma} v'_\mu \epsilon_\nu^* v_\rho \text{Tr} \{ O (1 + \not{v}) \gamma_\sigma (1 - \not{v}') \gamma_5 \} \right] \\
&\quad + \frac{1}{8\sqrt{3}} \frac{1}{\sqrt{v_o v'_o}} D_4^{(1/2)} \\
&\quad \times \left[ \text{Tr} \{ O (1 + \not{v}) \not{\epsilon}^* (1 + \not{v}') \} + i \epsilon^{\mu\nu\rho\sigma} v'_\mu \epsilon_\nu^* \text{Tr} \{ O (1 + \not{v}) \gamma_\rho \gamma_\sigma (1 - \not{v}') \gamma_5 \} \right] \\
\langle \tfrac{3}{2} 1^+ | O | 1 S_0 \rangle &= \frac{1}{4\sqrt{6}} \frac{1}{\sqrt{v_o v'_o}} \left( m_2 B^{(3/2)} + D_1^{(3/2)} + D_2^{(3/2)} \right) \\
&\quad \times \left[ -\text{Tr} \{ O (1 + \not{v}) (1 + \not{v}') \} (v \cdot \epsilon^*) + \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} v'_\mu \epsilon_\nu^* v_\rho \text{Tr} \{ O (1 + \not{v}) \gamma_\sigma (1 - \not{v}') \gamma_5 \} \right] \\
&\quad + \frac{1}{4\sqrt{6}} \frac{1}{\sqrt{v_o v'_o}} D_4^{(3/2)} \\
&\quad \times \left[ -\text{Tr} \{ O (1 + \not{v}) \not{\epsilon}^* (1 + \not{v}') \} + \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} v'_\mu \epsilon_\nu^* \text{Tr} \{ O (1 + \not{v}) \gamma_\rho \gamma_\sigma (1 - \not{v}') \gamma_5 \} \right] \\
\langle \tfrac{3}{2} 2^+ | O | 1 S_0 \rangle &= -\frac{1}{8\sqrt{3}} \frac{1}{\sqrt{v_o v'_o}} \left( m_2 B^{(3/2)} + D_1^{(3/2)} + D_2^{(3/2)} \right) \text{Tr} \{ O (1 + \not{v}) \gamma^\mu (1 - \not{v}') \gamma_5 \} v^\nu \epsilon_{\mu\nu}^* \\
&\quad - \frac{1}{8\sqrt{3}} \frac{1}{\sqrt{v_o v'_o}} D_4^{(3/2)} \text{Tr} \{ O (1 + \not{v}) \gamma^\mu \gamma^\nu (1 - \not{v}') \gamma_5 \} \epsilon_{\mu\nu}^*
\end{aligned}$$

### 3.2.3 Vector and axial matrix elements

Up to now, we did not take into account an explicit form for the current operator  $O$ . Let us see what becomes of the preceding relations when we use  $O = V_\lambda = \gamma_\lambda$  and  $O = A_\lambda = \gamma_\lambda \gamma_5$ .

$$\begin{aligned}
\langle \tfrac{1}{2} 0^+ | V_\lambda | 1 S_0 \rangle &= 0 \\
\langle \tfrac{1}{2} 0^+ | A_\lambda | 1 S_0 \rangle &= -\frac{1}{2\sqrt{3}} \frac{1}{\sqrt{v_o v'_o}} (v_\lambda - v'_\lambda) \left\{ (1 + v \cdot v') \left( m_2 B^{(1/2)} + D_1^{(1/2)} + D_2^{(1/2)} \right) + 3 D_4^{(1/2)} \right\} \\
\langle \tfrac{1}{2} 1^+ | V_\lambda | 1 S_0 \rangle &= \frac{1}{\sqrt{12}} \frac{1}{\sqrt{v_o v'_o}} \left[ (v \cdot \epsilon^*) v'_\lambda + (1 - v \cdot v') \epsilon_\lambda^* \right]
\end{aligned}$$

$$\begin{aligned} & \times \left\{ (1 + v \cdot v') \left( m_2 B^{(1/2)} + D_1^{(1/2)} + D_2^{(1/2)} \right) + 3 D_4^{(1/2)} \right\} \\ \langle \tfrac{1}{2} 1^+ | A_\lambda |^1 S_0 \rangle &= -\frac{i}{\sqrt{12}} \frac{1}{\sqrt{v_o v'_o}} \epsilon_{\lambda\sigma\rho\tau} v^\sigma \epsilon^{*\rho} v'^\tau \left\{ (1 + v \cdot v') \left( m_2 B^{(1/2)} + D_1^{(1/2)} + D_2^{(1/2)} \right) + 3 D_4^{(1/2)} \right\} \end{aligned}$$

for the  $j = 1/2$  multiplet, and

$$\begin{aligned} \langle \tfrac{3}{2} 1^+ | V_\lambda |^1 S_0 \rangle &= \frac{1}{2\sqrt{6}} \frac{1}{\sqrt{v_o v'_o}} \left[ (v \cdot \epsilon^*) \{ v'_\lambda (v \cdot v' - 2) - 3 v_\lambda \} + \epsilon_\lambda^* (1 - v \cdot v') (1 + v \cdot v') \right] \\ &\quad \times \left\{ m_2 B^{(3/2)} + D_1^{(3/2)} + D_2^{(3/2)} \right\} \\ \langle \tfrac{3}{2} 1^+ | A_\lambda |^1 S_0 \rangle &= -\frac{i}{2\sqrt{6}} \frac{1}{\sqrt{v_o v'_o}} (1 + v \cdot v') \epsilon_{\lambda\sigma\rho\tau} v^\sigma \epsilon^{*\rho} v'^\tau \left\{ m_2 B^{(3/2)} + D_1^{(3/2)} + D_2^{(3/2)} \right\} \\ \langle \tfrac{3}{2} 2^+ | V_\lambda |^1 S_0 \rangle &= \frac{i}{2} \frac{1}{\sqrt{v_o v'_o}} \epsilon_{\lambda\alpha\mu\beta} \epsilon^{*\mu\nu} v_\nu v^\alpha v'^\beta \left\{ m_2 B^{(3/2)} + D_1^{(3/2)} + D_2^{(3/2)} \right\} \\ \langle \tfrac{3}{2} 2^+ | A_\lambda |^1 S_0 \rangle &= \frac{1}{2} \frac{1}{\sqrt{v_o v'_o}} \left[ (1 + v \cdot v') \epsilon_{\lambda\nu}^* v^\nu - v^\mu v^\nu \epsilon_{\mu\nu}^* v'_\lambda \right] \left\{ m_2 B^{(3/2)} + D_1^{(3/2)} + D_2^{(3/2)} \right\} \end{aligned}$$

for the  $j = 3/2$  multiplet. As expected, all these matrix elements are manifestly covariant.

### 3.3 Isgur-Wise scaling

Looking at the results of the last section, it is fairly obvious that all the matrix elements can be written using two different quantities,  $\tau_{1/2}(v \cdot v')$  and  $\tau_{3/2}(v \cdot v')$ , following the notation introduced by Isgur and Wise in [5]. These  $\tau_j(v \cdot v')$ 's are:

$$\begin{aligned} \tau_{1/2}(v \cdot v') &= \frac{1}{2\sqrt{3}} \left\{ (1 + v \cdot v') \left( m_2 B^{(1/2)} + D_1^{(1/2)} + D_2^{(1/2)} \right) + 3 D_4^{(1/2)} \right\} \\ \tau_{3/2}(v \cdot v') &= \frac{1}{\sqrt{3}} \left\{ m_2 B^{(3/2)} + D_1^{(3/2)} + D_2^{(3/2)} \right\} \end{aligned} \quad (3.2)$$

Two remarks though: our definition of the state  $|\tfrac{1}{2} 1^+\rangle$  differs from the one given in [5] by an overall minus sign; and, secondly, we are using another normalization of the states. As a consequence, there is an overall factor of  $2\sqrt{m_P m_{D^{**}}} \sqrt{v_o v'_o}$  between our transition amplitudes and the transition amplitudes written in [5].

## 4 The Bjorken sum rule

Generally speaking, the Bjorken sum rule is a way of expressing duality (between quarks and hadrons), which, in the case of the heavy quark mass limit, is an exact duality. Mathematically, this rule reads [3]:

$$\begin{aligned} \bar{h}_{\mu\nu}(\vec{v}, \vec{v}') &\equiv \left( \sum_n \langle n, \vec{P}' | O^\mu | 0, \vec{P} \rangle \times \langle n, \vec{P}' | O^\nu | 0, \vec{P} \rangle^* \right)_{\text{where } m_1 \rightarrow \infty} \\ &= \frac{1}{2} \sum_{s_1, s'_1} [\bar{u}_{s'_1}(\vec{v}') O^\mu u_{s_1}(\vec{v})] [\bar{u}_{s'_1}(\vec{v}') O^\nu u_{s_1}(\vec{v})]^* \equiv \bar{h}_{\mu\nu}^{\text{free quark}} \end{aligned} \quad (4.1)$$

where  $\bar{h}_{\mu\nu}(\vec{v}, \vec{v}')$  is the “hadronic tensor” which describes the transition between an initial meson state to all possible meson states containing a heavy quark, and where  $\bar{h}_{\mu\nu}^{\text{free quark}}$  is the analog for free quarks. Note that  $n$  is used to label the vectors of the *complete* basis  $\{|n, \vec{P}\rangle\}$ .

By using the expressions of the current matrix elements which are involved in the definition of  $\bar{h}_{\mu\nu}(\vec{v}, \vec{v}')$  and expanding around  $w = \vec{v} \cdot \vec{v}' = 1$ , we get the new following Bjorken sum rule:

$$\sum_n \left( |\tau_{1/2}^{(n)}(1)|^2 + 2 |\tau_{3/2}^{(n)}(1)|^2 \right) = \rho^2 - \frac{1}{4} \quad (4.2)$$

where  $\rho^2$  is the slope of the elastic ground state Isgur-Wise scaling function and where the superscripts ( $n$ ) characterize the radial excitations of the P wave states, to which the results of the preceding section obviously apply (these ( $n$ ) correspond exactly to the labels  $n$  used in (4.1)). In [3], the physical meaning and a detailed demonstration of that sum rule have been given. In the following, we are just going to verify that (4.2) still holds with the covariant formalism.

#### 4.1 The $\tau_j(1)$ 's

Checking (4.2) begins with evaluating the  $\tau_j$ 's at  $v \cdot v' = 1$ . Since they are given in a covariant way, let's choose the particular frame of reference where:

$$\left| \begin{array}{ll} \vec{v}' = \vec{0} & (\text{then } v'_0 = 1) \\ \vec{v} \neq \vec{0} \text{ with } \vec{v} \text{ small} & (\text{then } v_0 \simeq 1 + \frac{1}{2} \vec{v}^2) \end{array} \right.$$

Therefore, to the second order in  $\vec{v}$ , we get:

$$\begin{aligned} p \cdot v' &= p_0 \\ p \cdot v &\simeq p_0 - \vec{p} \cdot \vec{v} + \frac{1}{2} p_0 \vec{v}^2 \\ v \cdot v' &\simeq 1 + \frac{1}{2} \vec{v}^2 \end{aligned}$$

and also

$$\begin{aligned} \|\vec{B}_{v'}^{-1} p\|^2 &= \vec{p}^2 \\ \|\vec{B}_v^{-1} p\|^2 &\simeq \vec{p}^2 - 2 p_0 \vec{p} \cdot \vec{v} \end{aligned}$$

Then, we substitute these expressions into (3.2) and expand to the first order in  $\vec{v}$  the  $\phi$  functions of (3.2). Owing to rotational invariance, the terms of the form  $\vec{v}/\vec{v}^2$  vanish and, when we take  $\vec{v}$  equal to zero, we are left with:

$$\tau_j^{(n)}(1) = \int \frac{p^2 dp}{(2\pi)^2} \phi_j^{*(n)}(p^2) F_j(p^2) \quad (4.3)$$

with  $p = \|\vec{p}\|$  and where

$$\begin{aligned} F_{1/2}(p^2) &= -\frac{1}{3\sqrt{3}} \left\{ \varphi(p^2) \frac{p^2}{m+p_0} \left( 3 + \frac{m}{p_0} \right) + 4 \frac{d\varphi}{dp^2}(p^2) p_0 p^2 \right\} \\ F_{3/2}(p^2) &= -\frac{1}{3\sqrt{3}} \left\{ \varphi(p^2) \frac{p^2}{m+p_0} \frac{m}{p_0} + 4 \frac{d\varphi}{dp^2}(p^2) p_0 p^2 \right\} \end{aligned}$$

#### 4.2 Connection between the $F_j$ 's and the sum of the $\tau_j$ 's

Let us work in the part  $\mathcal{H}_{\vec{j}}$  of the total Hilbert space  $\mathcal{H}$  describing the wave functions, that is, let us fix the spin of the heavy quark (which we know is irrelevant). Then, the partial wave function writes:

$$\Psi_j^{(n)\mu}(\vec{p}) = \sum_m \Phi_j^{(n)m}(\vec{p}) \langle 1 \ m \ 1/2 \ \mu - m \mid j \ \mu \rangle \chi^{\mu-m}$$

In  $\mathcal{H}_{\vec{j}}$ , the *closure relation* reads:

$$\sum_{j,n,\mu} \Psi_j^{(n)\mu\dagger}(\vec{p}) \Psi_j^{(n)\mu}(\vec{p}') = (2\pi)^3 \delta(\vec{p} - \vec{p}') \quad (4.4)$$

If we multiply (4.4) by  $\int d\Omega d\Omega' Y_1^{m_1}(\Omega) Y_1^{m'_1}(\Omega')$  and sandwich the result between  $\chi^\nu$  on the left and  $\chi^{\nu'\dagger}$  on the right, we obtain:

$$\sum_{j,n} \frac{4\pi}{3} p p' \phi_j^{*(n)}(p^2) \phi_j^{(n)}(p'^2) \delta_{\mu\mu'} \langle 1 \ m_1 \ 1/2 \ \nu \mid j \ \mu \rangle \langle 1 \ m'_1 \ 1/2 \ \nu' \mid j \ \mu' \rangle = (2\pi)^3 \frac{\delta(p-p')}{p'^2} \delta_{m_1 m'_1} \delta_{\nu \nu'}$$

where  $p = \|\vec{p}\|$  and  $p' = \|\vec{p}'\|$ . Finally, by multiplying this last relation by  $\langle 1 \ m_1 \ 1/2 \ \nu \mid J \ M \rangle \langle 1 \ m'_1 \ 1/2 \ \nu' \mid J' \ M' \rangle$  and summing the result over  $m_1, m'_1, \nu, \nu'$ , we get:

$$\sum_n \phi_j^{*(n)}(p^2) \phi_j^{(n)}(p'^2) = 6\pi^2 \frac{\delta(p-p')}{pp'^3}$$

for each value of  $j$ . We now have all the pieces to calculate  $\sum |\tau_j|^2$ ; starting from (4.3) and summing its squared norm, we get:

$$\sum_n |\tau_j^{(n)}(1)|^2 = \frac{3}{8\pi^2} \int dp |F_j(p^2)|^2$$

### 4.3 Checking the Bjorken sum rule for $\rho^2$

We now are able to evaluate the left hand side of (4.2). After integrating by parts (the integral of the divergence vanishes), we get:

$$\sum_n |\tau_{1/2}^{(n)}(1)|^2 + 2 \sum_n |\tau_{3/2}^{(n)}(1)|^2 = \frac{2}{3} \int \frac{dp}{(2\pi)^2} \left\{ \varphi(p^2) \varphi(p^2)^* \left[ \frac{1}{4} \frac{p^4}{p_0^2} - \frac{p_0 + 2m}{p_0 + m} p^2 \right] + \frac{d\varphi}{dp^2}(p^2) \frac{d\varphi}{dp^2}(p^2)^* 4 p_0^2 p^4 \right\}$$

Regarding the right hand side of (4.2), we start from the expression of the slope  $\rho^2$  given by the equation (29) of [1]:

$$\rho^2 = \frac{1}{3} \int \frac{d\vec{p}}{(2\pi)^3} [\vec{\nabla} p^0 \varphi(\vec{p})]^* \cdot [\vec{\nabla} p^0 \varphi(\vec{p})] + \int \frac{d\vec{p}}{(2\pi)^3} \left[ \frac{2}{3} + \frac{1}{4} \frac{m^2}{(p^0)^2} - \frac{1}{3} \frac{m}{p^0 + m} \right] \varphi(\vec{p})^* \varphi(\vec{p})$$

Then, we use

$$\vec{\nabla}(p_0 \varphi) = p_0 \vec{\nabla} \varphi + \varphi \vec{\nabla} p_0 = 2 p_0 \vec{p} \frac{d\varphi}{dp^2} + \frac{\vec{p}}{p_0} \varphi$$

and, after another integration by parts, we get

$$\rho^2 - \frac{1}{4} = \frac{2}{3} \int \frac{dp}{(2\pi)^2} \left\{ \varphi(p^2) \varphi(p^2)^* \left[ \frac{1}{4} \frac{p^4}{p_0^2} - \frac{p_0 + 2m}{p_0 + m} p^2 \right] + \frac{d\varphi}{dp^2}(p^2) \frac{d\varphi}{dp^2}(p^2)^* 4 p_0^2 p^4 \right\}$$

showing that (4.2) is indeed verified.

## 5 Conclusion

In this paper, we have considered the  $B \rightarrow D^{**}$  type transitions quark models à la BT in the heavy quark limit. As already shown for the  $B \rightarrow D^{(*)}$  transitions [1], these models verify covariance and heavy quark symmetry. Consequently, all the hadronic matrix elements between a pseudoscalar state and a P-wave meson state, both containing a quark with infinite mass, can be expressed using two different functions, namely  $\tau_{1/2}$  and  $\tau_{3/2}$ , which we have computed in these models:

$$\begin{aligned} \tau_{1/2}(w) = & \frac{1}{2\sqrt{3}} \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{(p_2 \cdot v')(p_2 \cdot v)}}{\sqrt{(p_2 \cdot v' + m_2)(p_2 \cdot v + m_2)}} \phi_{\frac{1}{2}}((p_2 \cdot v')^2 - m_2^2)^* \varphi((p_2 \cdot v)^2 - m_2^2) \\ & \times \frac{(p_2 \cdot v)(p_2 \cdot v' + m_2) - (p_2 \cdot v')(p_2 \cdot v + m_2) + (1 - v \cdot v') m_2^2}{1 - v \cdot v'} \end{aligned}$$

and

$$\begin{aligned} \tau_{3/2}(w) = & \frac{1}{\sqrt{3}} \frac{1}{1 - (v \cdot v')^2} \int \frac{d\vec{p}_2}{(2\pi)^3} \frac{1}{p_2^0} \frac{\sqrt{(p_2 \cdot v')(p_2 \cdot v)}}{\sqrt{(p_2 \cdot v' + m_2)(p_2 \cdot v + m_2)}} \phi_{\frac{3}{2}}((p_2 \cdot v')^2 - m_2^2)^* \varphi((p_2 \cdot v)^2 - m_2^2) \\ & \times \left\{ \frac{3}{2} \frac{1}{1 + v \cdot v'} [p_2 \cdot (v + v')]^2 - (p_2 \cdot v)(2 p_2 \cdot v' - m_2) - (p_2 \cdot v') [p_2 \cdot v + (v \cdot v') m_2] - \frac{1 - v \cdot v'}{2} m_2^2 \right\} \end{aligned}$$

In the two last equations the apparent poles at  $v \cdot v' = 1$  are canceled by zeros in the numerators.

We have also checked the validity of the Bjorken sum rule already demonstrated in [3]. These results are general for all quark models à la BT, independently of the precise dynamics. To complete the calculation, it is necessary to have an explicit form for the radial functions  $\phi_j$  and  $\varphi$ . Therefore, what remains to be done is to take a hamiltonian and then use it to solve a rest-frame Schrödinger-type equation, leading to wave functions from which the radial parts can be extracted. That will be done in a forthcoming paper. Of course, all the difficulties lie in the choice of the hamiltonian describing the meson (as an example, the resulting values of  $\rho^2$  must not be too high<sup>4</sup>). When this is achieved, we will be able to compute, consistently with the Bjorken sum rule,  $\rho^2$  and the rate of the semileptonic decay  $B \rightarrow D^{**} l \nu$ .

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<sup>4</sup>The experimental value of  $\hat{\rho}^2 = 0.87 \pm 0.12 \pm 0.08$  from CLEO II is not easy to compare to  $\rho^2$  in our model since  $1/m_1$  corrections and radiative corrections should be considered. Still we would like to avoid a too large discrepancy.